Tutorial 4 : Selected problems of Assignment 4

Leon Li

2/10/2019

Q1) (Ex 4, Q3) Show that the generalised Hölder's inequality holds:

For any $n \geq 2$, for any n integrable functions $f_1, \cdots, f_n : [a,b] \to \mathbb{C}$,

for all $p_1, \cdots, p_n > 1$ such that $\frac{1}{p_1} + \cdots + \frac{1}{p_n} = 1$.

$$\int_a^b |f_1 f_2 \cdots f_n| \leq \left(\int_a^b |f_1|^{p_1}\right)^{\frac{1}{p_1}} \cdots \left(\int_a^b |f_n|^{p_n}\right)^{\frac{1}{p_n}}$$

Sol) Prove by induction on $n \geq 2$:

Base step $(n=2)$: True, by the usual Hölder's inequality for functions:

$\int_a^b |fg| \leq \left(\int_a^b |f|^p\right)^{\frac{1}{p}} \left(\int_a^b |g|^q\right)^{\frac{1}{q}}$ (Proof: See Lecture note Ch. 2 Thm 2.10)

Inductive Step: Suppose true for $n=N$, showing true for $n=N+1$:

given $f_1, \cdots, f_{N+1}$ as above. $p_1, \cdots, p_{N+1} > 1$ such that $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_{N+1}} = 1$

By Base step applying to $f = f_1$; $g = f_2 \cdots f_{N+1}$, $p = p_1$, $q = \frac{1}{\frac{1}{p_2} + \cdots + \frac{1}{p_{N+1}}}$

$\text{LHS} = \int_a^b |f_1||f_2 \cdots f_{N+1}| \leq \left(\int_a^b |f_1|^p\right)^{\frac{1}{p}} \left(\int_a^b |f_2 \cdots f_{N+1}|^q\right)^{\frac{1}{q}}$

By inductive step applying to $\tilde{f}_1 = f_2^q, \cdots, \tilde{f}_N = f_{N+1}^q$, $\tilde{p}_1 = \frac{p_2}{q}, \cdots, \tilde{p}_N = \frac{p_{N+1}}{q}$:

$\int_a^b |f_2^q \cdots f_{N+1}^q| \leq \left(\int_a^b |f_2|^{p_2}\right)^{\frac{q}{p_2}} \cdots \left(\int_a^b |f_{N+1}|^{p_{N+1}}\right)^{\frac{q}{p_{N+1}}}$

$\therefore \left(\int_a^b |f_1|^p\right)^{\frac{1}{p}} \left(\int_a^b |f_2 \cdots f_{N+1}|^q\right)^{\frac{1}{q}} \leq \left(\int_a^b |f_1|^{p_1}\right)^{\frac{1}{p_1}} \left(\int_a^b |f_2|^{p_2}\right)^{\frac{1}{p_2}} \cdots \left(\int_a^b |f_{N+1}|^{p_{N+1}}\right)^{\frac{1}{p_{N+1}}} = \text{RHS}$

Q2) (Ex.4, Q10) Fix $1 \leq p < +\infty$, define $\ell^p := \{(a_n)_{n=1}^{\infty} \mid a_n \in \mathbb{R} ; \sum_{n=1}^{\infty} |a_n|^p < +\infty\}$

and define p-norm $\|\cdot\|_p : \ell^p \to \mathbb{R}$ by $\|a\|_p := \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}}$

Show that $(\ell^p, \|\cdot\|_p)$ is a normed space.

Sol) Check the axioms [N1]-[N3] for normed spaces:

[N1]: $\forall a \in \ell^p$, $\|a\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}} \geq 0$, which

$(=0)$ holds $\Leftrightarrow \forall n, \ |a_n| = 0 \Leftrightarrow a = 0$

[N2]: $\forall a \in \ell^p$, $\forall \alpha \in \mathbb{R}$, $\|\alpha a\|_p = \left(\sum_{n=1}^{\infty} |\alpha a_n|^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty} |\alpha|^p |a_n|^p\right)^{\frac{1}{p}}$

$= |\alpha| \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}} = |\alpha| \|a\|_p$

[N3]: $\forall a, b \in \ell^p$, $\forall N \in \mathbb{N}$, define $a^{(N)} = (a_1, \cdots, a_N)$, $b^{(N)} = (b_1, \cdots, b_N) \in \mathbb{R}^N$

$\left(\sum_{n=1}^{N} |a_n + b_n|^p\right)^{\frac{1}{p}} = \|a^{(N)} + b^{(N)}\|_p \leq \|a^{(N)}\|_p + \|b^{(N)}\|_p$ (by Minkowski inequality on $(\mathbb{R}^N, \|\cdot\|_p)$
see also Note 2, Example 2.3)

$= \left(\sum_{n=1}^{N} |a_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{N} |b_n|^p\right)^{\frac{1}{p}} \leq \|a\|_p + \|b\|_p$

$\therefore$ Take $N \to +\infty$: $\|a+b\|_p \leq \|a\|_p + \|b\|_p$

$\therefore (\ell^p, \|\cdot\|_p)$ is a normed space.

Q3) (Ex. 4, Q7) Under same notations as in Q2,

show that for any $0 < p < 1$, $(\ell^p, \|\cdot\|_p)$ is NOT a normed space.

Sol) Showing [N3] is false: Choose $a = (1, 0, \cdots)$; $b = (0, 1, 0, \dots)$,

then $\|a\|_p = 1 = \|b\|_p$; $\|a+b\|_p = (1^p + 1^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}$

$\because$ $0 < p < 1 \Rightarrow \|a+b\|_p = 2^{\frac{1}{p}} > 2 = \|a\|_p + \|b\|_p$

$\therefore$ [N3] is false, hence $(\ell^p, \|\cdot\|_p)$ is NOT a normed space.

Rmk Exactly the same argument shows that for any $n \geq 2$, $0 < p < 1$,

$(\mathbb{R}^n, \|\cdot\|_p)$ is NOT a normed space.